Quartic points on the Fermat quartic over a quadratic extension of $\mathbb{Q}(\sqrt{2})$

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Quadratic points on the Fermat quartic

$$x^4 + y^4 = 1 \tag{F_4}$$

Theorem (Aigner 1934, Mordell 1967)

The only quadratic points on F_4 are

$$(\pm i, 0),\ (0, \pm i),\ \left(\epsilon_1 \frac{1 + \epsilon\sqrt{-7}}{2}, \epsilon_2 \frac{1 - \epsilon\sqrt{-7}}{2}\right),$$

where $\epsilon^2 = \epsilon_1^2 = \epsilon_2^2 = 1$.

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All points on F_4 lying in a quadratic extension of $\mathbb{Q}(\zeta_8) = \mathbb{Q}(i, \sqrt{2})$ lie in one of $\mathbb{Q}(i, \sqrt{2})$, $\mathbb{Q}(\sqrt[4]{2}, \zeta_8)$, $\mathbb{Q}(\zeta_3, \zeta_8)$, $\mathbb{Q}(\sqrt{-7}, \zeta_8)$.

- The authors study the 4-torsion points on the Jacobian of F_4
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Extending Mordell

- Let K be a quadratic extension of $L = \mathbb{Q}(\sqrt{2})$
- Suppose

$$x^4 + y^4 = 1, \qquad x, y \in K$$

- Let

$$t = \frac{1 - x^2}{y^2}.$$

- We can rearrange to get

$$x^2 = \frac{1 - t^2}{1 + t^2}, \qquad y^2 = \frac{2t}{1 + t^2}.$$

- Either $t \in L$ or $t \in K$

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$$t = \frac{1-x^2}{y^2}, \qquad x^2 = \frac{1-t^2}{1+t^2}, \qquad y^2 = \frac{2t}{1+t^2}$$

- Either $x \in L$, $y \in L$ or $x/y \in L$
- If $x \in L$ then $u^2 = (1-t^2)(1+t^2)$, $\qquad u \in L$
- This is isomorphic to the elliptic curve E with Cremona label 32a1
- Taking the pre-image of points in $E(L) = E(\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ gives us points on F_4 over $\mathbb{Q}$ and $\mathbb{Q}(i)$
- Similar computations in the other cases give us points on F_4 over $\mathbb{Q}(\sqrt[4]{2})$ and $\mathbb{Q}(i\sqrt[4]{2})$

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Extending Mordell

- Suppose $t \in K, t \notin L$
- Let $minpol_L(t) = t^2 + \beta t + \gamma, \quad \beta, \gamma \in L$
- Let $A = (1 + t^2)xy$. We can also write

$$A = \lambda + \mu t, \quad \lambda, \mu \in L$$

- We can square both expressions for A:

$$(\lambda + \mu t)^2 - 2t(1 - t^2) = minpol_L(t)(\rho + \sigma t) \tag{1}$$

- We get a point $(\lambda + \mu(-\rho/\sigma), -\rho/\sigma)$ on the elliptic curve

$$E: \; Y^2 = -2X^3 + 2X$$

defined over L; E is the elliptic curve with Cremona label 64a1!

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Extending Mordell

- Squaring and equating the above leads to

$$(\lambda + \mu t)^2 - 2t(1 - t^2) = minpol_L(t)(\rho + \sigma t) \qquad (2)$$

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- Let $B = (1 + t^2)y$. We similarly get a L–point on the elliptic curve with Cremona label 32a1
- Recall that both elliptic curves have finite rank over L
- This gives us finitely many possibilities for t hence finitely many equations to solve
- Solving these equations gives us points on F_4 defined over $\mathbb{Q}(\sqrt{2}, \sqrt{-7})$ and $\mathbb{Q}(\sqrt{2}, i)$

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More points on the Fermat quartic?

- The key in Mordell's proof: the elliptic curves `32a1` and `64a1` have rank 0 over $\mathbb{Q}$
- These elliptic curves have rank 0 over $\mathbb{Q}(\sqrt{2})$
- Let L be a number field such that the elliptic curves have rank 0 over L
- Can use Mordell's strategy to find all points on F_4 lying in a quadratic extension of L

Other quartic points on the Fermat quartic

- Pedro-José Cazorla Garcia pointed out that

$$(\sqrt{3})^4 + 2^4 = (\sqrt{5})^4$$

- The elliptic curve 32a1 has rank 0 over $\mathbb{Q}(\sqrt{3})$ and the elliptic curve 64a1 has rank 1 over $\mathbb{Q}(\sqrt{3})$
- The elliptic curve 32a1 has rank 1 over $\mathbb{Q}(\sqrt{5})$ and the elliptic curve 64a1 has rank 0 over $\mathbb{Q}(\sqrt{5})$
- Mordell's strategy won't work here!

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Thanks for listening :)